## Notation and Relevant Relations

Let "$(X \,|\, c\,)$" represent eventuality $X$ in the context of condition $c$. $X$ may be said to be *potential* (though impossibilities are not disallowed), while $c$ may be said to be *given* as evidence or by *presupposition*.[1]

Axiomatic systems often begin with a formally undefined relation of *weak supraprobability* (or perhaps with *weak infraprobability*), which allows an elegance but has fostered some confusions of interpretation. I hope that the reader will forgive my beginning differently. Let "$\rhd$" represent *strict supraprobability*, let "$\boxdot$" represent *equiprobability*, and define "weak supraprobability"

$$\unrhd \stackrel{\text{def}}{=} (\rhd \cup \boxdot) \;. \tag{D1}$$

We'll also have some use for the following[2]

$$\lhd \stackrel{\text{def}}{=} \rhd^{-1} \;, \tag{D2}$$

$$\unlhd \stackrel{\text{def}}{=} \unrhd^{-1} \;, \tag{D3}$$

$$\odot \stackrel{\text{def}}{=} \overline{\rhd \cup \boxdot \cup \rhd^{-1}} \;. \tag{D4}$$

The relation $\odot$ is empty when $\unrhd$ is a complete preördering.

The first four axiomata do not invite much controversy in their content, and elsewhere might be treated as if definitional.

**Antisymmetry of Strict Supraprobability:**

$$\overline{\left(\begin{array}{c} [(X \,|\, c_1\,) \rhd (Y \,|\, c_2\,)] \\ \wedge \\ [(Y \,|\, c_2\,) \rhd (X \,|\, c_1\,)] \end{array}\right)} \forall\,(X, Y, c_1, c_2) \tag{A1}$$

**Reflexivity and Symmetry of Equiprobability:**

$$\left[\begin{array}{c} [(X \,|\, c_1\,) \boxdot (X \,|\, c_1\,)] \\ \wedge \\ \left(\begin{array}{c} [(X \,|\, c_1\,) \boxdot (Y \,|\, c_2\,)] \\ \Leftrightarrow \\ [(Y \,|\, c_2\,) \boxdot (X \,|\, c_1\,)] \end{array}\right) \end{array}\right] \forall\,(X, Y, c_1, c_2) \tag{A2}$$

**Mutual Exclusion of Strict Supraprobability and Equiprobability:**

$$\overline{[(X \,|\, c_1\,) \rhd (Y \,|\, c_2\,)] \wedge [(X \,|\, c_1\,) \boxdot (Y \,|\, c_2\,)]} \forall\,(X, Y, c_1, c_2) \tag{A3}$$

[1] The use of lowercase letters for things not considered as eventualities is simply a convenience; where "$(X \,|\, c\,)$" is meaningful, so would be "$(c \,|\, X\,)$".

[2] Throughout, an overscore will represent *complementation* — logical negation of propositions, set complementation in the case of relations.

**Transitivity of Weak Supraprobability:**

$$\left[\begin{pmatrix} [(X\,|\,c_1\,) \trianglerighteq (Y\,|\,c_2\,)] \\ \wedge \\ [(Y\,|\,c_2\,) \trianglerighteq (Z\,|\,c_3\,)] \end{pmatrix} \\ \Rightarrow \\ [(X\,|\,c_1\,) \trianglerighteq (Z\,|\,c_3\,)]\right] \forall\,(X,Y,Z,c_1,c_2,c_3) \tag{A4}$$

## Interpretation

### The Subjects of the Relations

Any event corresponds to a proposition that the event has occurred, and any proposition corresponds to the event that the proposition is true. Hereïn, I will treat the $X$ and $c$ of $(X\,|\,c\,)$ as themselves propositions, and an expression such as "$(X_1 \vee X_2\,|\,c_1 \wedge c_2\,)$" as meaningful.[3] (Any unconditional probabilities would be equivalent to those whose conditions were taken to be known with certainty, as in the case of tautologies.) Were these instead events, the equivalent expression would simply be "$(X_1 \cup X_2\,|\,c_1 \cap c_2\,)$". Likewise, an expression such as "$c \Rightarrow X$", representing an eventuality as logically implied by a context, would have an equivalent "$c \subseteq X$".

### The Weak Relations

*Probability* has been equated or identified with various things and with conflations or confusions amongst them — idiosyncratic confidence, betting quotient, logical state intermediate between truth and falsehood, propensity (a generalization of causation) with its inverse, combinatoric distribution, and frequency. The formal relations above might be interpretted to fit most or all of these notions, but it is worth mention that these relations cannot each describe a positive state of belief,[4] nor could they if the definitions were along the lines of

$$\left[\rhd \stackrel{\text{def}}{=} (\trianglerighteq \setminus \trianglelefteq)\right] \wedge \left[\lhd \stackrel{\text{def}}{=} (\trianglelefteq \setminus \trianglerighteq)\right] \\ \wedge \\ \left[\boxdot \stackrel{\text{def}}{=} (\trianglerighteq \cap \trianglelefteq)\right]$$

If "$(X\,|\,c\,) \trianglerighteq (Y\,|\,d\,)$" and "$(Y\,|\,d\,) \trianglelefteq (X\,|\,c\,)$" represented having *some* relative confidence and *being no less confident* in $(X\,|\,c\,)$ than in $(Y\,|\,d\,)$ *without* being either more confident in $(X\,|\,c\,)$ (which would preclude equal confidence) or being

[3]It might be noted that

$$\begin{pmatrix} (c \Rightarrow X_2) \\ \Rightarrow \\ [(X_1\,|\,c\,) = (X_1 \wedge X_2\,|\,c\,)] \end{pmatrix} \forall\,(X_1,X_2,c)\ .$$

[4]The probability relations $\rhd$, $\boxdot$, $\trianglerighteq$, and $\odot$ all may represent *states of belief*, but *states of belief* may entail what is believed, what is rejected, and what is neither believed nor rejected.

equally confident in the two (which would preclude being more confident in the one), then

$$\begin{array}{c}[(X\,|\,c)\trianglerighteq(Y\,|\,d)]\\ \wedge\\ \overline{(X\,|\,c)\triangleright(Y\,|\,d)}\wedge\overline{(X\,|\,c)\boxdot(Y\,|\,d)}\end{array}$$

would be somehow possible, in contradiction to the formal definition of "$\trianglerighteq$".

While a claim of form "$(X\,|\,c)\trianglerighteq(Y\,|\,d)$" does not seem peculiarly problematic as a *premise* under the aforementioned various notions of *probability*, it should cause some concern if an axiomatic system allows a claim of this form to result as a conclusion when none of the premises contain such a form (or its equivalent). In the particular case of subjectivism, it seems self-alienated to *conclude* merely that $(X\,|\,c)\trianglerighteq(Y\,|\,d)$. And if, taken as a whole, a system of axiomata for subjective probability cannot proceed past a conclusion of such form, then one should ask by what justification it has excluded $(X\,|\,c)\odot(Y\,|\,d)$; a conclusion that $(X\,|\,c)\trianglerighteq(Y\,|\,d)$ requires a commitment to exactly one of the basic relations ($\triangleright$ and $\boxdot$) without itself being able to make such a commitment.

**Noncomparability**

The relation $\odot$ holds between pairs of contextualized eventualities that are not related by $\trianglerighteq$ nor by $\trianglelefteq$. $\odot$ thus implies the *absence* of a relative probability. A *pure* frequentism or a *pure* combinatoric interpretation of probability would force $\odot$ to be empty, as also (trivially) would betting quotients. But logicism and subjectivism allow it to be non-empty. (An *impure* frequentism would be one in which $\trianglerighteq$ and $\trianglelefteq$ were always about beliefs about frequency; similarly for an impure combinatoric interpretation.)

**Narrowed Noncomparability** There may be some temptation to write something such as

$$[(X_1\,|\,c_1)\odot(X_2\,|\,c_2)]\wedge\overline{(X_1\,|\,c_1)\,R\,(X_2\,|\,c_2)}\,, \tag{1}$$

where $R\in\{\triangleright,\boxdot,\triangleleft\}$, to capture the notion that the probability of $(X_1\,|\,c_1)$ relative to $(X_2\,|\,c_2)$ is unestablished but narrowed to two possibilities. However, the expression (1) does not capture this sort of limited exclusion; in fact, it contains a redundancy as, *ex definitione*,

$$[(X_1\,|\,c_1)\odot(X_2\,|\,c_2)]\Leftrightarrow\left[\begin{array}{c}\overline{(X_1\,|\,c_1)\triangleright(X_2\,|\,c_2)}\\ \wedge\\ \overline{(X_1\,|\,c_1)\boxdot(X_2\,|\,c_2)}\\ \wedge\\ \overline{(X_1\,|\,c_1)\triangleleft(X_2\,|\,c_2)}\end{array}\right].$$

The exclusion of $R$ may be written as

$$\begin{array}{c}[(X_1\,|\,c_1)\odot(X_2\,|\,c_2)]\vee[(X_1\,|\,c_1)\,S\,(X_2\,|\,c_2)]\vee[(X_1\,|\,c_1)\,T\,(X_2\,|\,c_2)]\\ \ni\\ \{S,T\}\equiv(\{\triangleright,\boxdot,\triangleleft\}\setminus\{R\})\end{array}. \tag{2}$$

In considering a case in which it is held that one of the three relations $\rhd$, $\boxdot$, $\lhd$ may be ruled-out but either of the other two remain a possibility, it is important not to confuse the interpretation of these relations as themselves beliefs with beliefs about these relations. If we are describing, for example, an agent who is sure that two things are not equiprobable but unsure as to which is supraprobable, then we are employing a notion of *probability* as something external to the consciousness of the agent.

## Remaining Axiomata

Though the various candidates offered as the fundamental notion or notions of probability are intertangled, the justification for any ostensible axiom would vary significantly depending upon which candidate or candidates were selected, if indeed any candidate were selected.

Even to the extent that axiomata (A1)–(A4) might appear to be purely definitional, there is a question of why relations with these properties *apply* to what is of interest (knowledge, belief, frequency, or whatever). Frequentism could find $\rhd$ in $>$ and $\boxdot$ in $=$ amongst arithmetic ratios. The ostensibly intuitive logicisms of Keynes and of Koopman would claim an immediate apprehension of *supraprobability*, strong or weak [4][5][6]; a more humble logicism could claim these relations to be somehow discovered in experience. Subjectivism might claim to find them rather directly in self-examination, or by contemplating inclinations of choice [7].

Without here embracing a position on the essence of probability, I would justify proposed axiomata as conforming across interpretations and as capturing important properties in a relatively simple fashion. From the perspective of some philosophical positions, these propositions would be regarded as no better than that (if not worse).

But logicists and subjectivists might take a different view. Some readers may think some or all of these propositions to be known intuitively. In that context, I note that the axiomata of *Subdivision*, of *Composition*, and of *Decomposition* offered below are both less complex and more general in application than those offered by Koopman [5][6]. (These differences are especially pronounced in the case of the principle of *Subdivision.*) Others might regard the axiomata hereïn as discovered principles of reasoning or as rationality constraints upon belief.

Although some of the individual axiomata are simpler than are corresponding principles in other systems, this system as whole is markedly more complicated than are familiar formal systems, in order that it be useable without any presumption of completeness.

Axiomata (A1)–(A4) treat the subjects of the probability relations as if themselves black boxes. The remaining axiomata concern how logical relations between what is given and contingencies and amongst whaty is given or amongst contingencies effect or affect probability relations.

**Axiom 5 (Generalized Complementarity):**

$$\left(\begin{array}{c}[(X\,|\,c_1\,)\trianglerighteq(Y\,|\,c_2\,)]\\\Rightarrow\\\left[\left(\overline{Y}\,|\,c_2\,\right)\trianglerighteq\left(\overline{X}\,|\,c_1\,\right)\right]\end{array}\right)\forall\,(X,Y,c_1,c_2) \tag{A5}$$

Koopman calls this principle "Antisymmetry" [5][6], but it is not what would normally be meant in claiming that $\trianglerighteq$ were antisymmetrical. (A5) may be seen as a variation on

$$\left(\begin{array}{c}[(X\,|\,c\,)\triangleright(Y\,|\,c\,)]\\\Rightarrow\\\left[\left(\overline{Y}\,|\,c\right)\triangleright\left(\overline{X}\,|\,c\,\right)\right]\end{array}\right)\forall\,(X,Y,c)\ , \tag{3}$$

which is often suggested as an axiom [3].

**Axiom 6 (Implication of Presupposition):**

$$\left(\begin{array}{c}(c_1\Rightarrow X)\\\Leftrightarrow\\{[(X\,|\,c_1\,)\trianglerighteq(Y\,|\,c_2\,)]}\end{array}\right)\forall\,(X,Y,c_1,c_2) \tag{A6}$$

(A6) makes no allowance for *unrecognized implications*. It might be practicable to do so with a more complicated proposition. (The lack of such allowance may be a significant problem if probability relations are interpretted as *subjective*;[5] there should be no crisis or even surprise in the thought that one may be mistaken about objective relations.) In any case, this axiom implies that implications of whatever is *given* are *maximally* probable.

From (A5) and (A6), one may conclude tha

$$\left(\begin{array}{c}(c_1\Rightarrow X)\\\Rightarrow\\\left[(Y\,|\,c_2\,)\trianglerighteq\left(\overline{X}\,|\,c_1\,\right)\right]\end{array}\right)\forall\,(X,Y,c_1,c_2)\ , \tag{4}$$

which could be adopted instead of (A6), to essentially the same final effect.

**Axiom 7 (Presuppositional Interposition):**

$$\left(\begin{array}{c}[(X\,|\,c_1\,)\trianglerighteq(X\,|\,c_1\vee c_2\,)]\\\Leftrightarrow\\{[(X\,|\,c_1\vee c_2\,)\trianglerighteq(X\,|\,c_2\,)]}\end{array}\right)\forall\,(X,c_1,c_2)\ \textbf{;} \tag{A7}$$

that is to say that if one context by itself makes an outcome more probable than the joint possibility of that context and another, then that other context by itself leaves that outcome less possible than the joint possibility, and *vice versa*.

[5]In an event-oriented framework, as favored by many subjectivists, the issue would of course be cases of sets not recognized as having a relation of form $c_1 \subseteq X$.

In a case in which $c_1 = (c_3 \wedge c_4)$ and $c_2 = (c_3 \wedge \overline{c_4})$,

$$\left(\begin{array}{c} [(X\,|\,c_3 \wedge c_4\,) \trianglerighteq (X\,|\,c_3\,)] \\ \Leftrightarrow \\ [(X\,|\,c_3\,) \trianglerighteq (X\,|\,c_3 \wedge \overline{c_4}\,)] \end{array}\right) \forall\,(X, c_3, c_4)\ . \tag{5}$$

If one used (5) as the axiom, then (A7) could be derived from it by nearly the reverse process, but noting also that the implication in (A7) is trivially true when $c_2 = c_1$.

Many readers will recognize that

$$\left[\begin{array}{c} [(X\,|\,c_1\,) \trianglerighteq (X\,|\,c_2\,)] \\ \Rightarrow \\ \left(\begin{array}{c} [(X\,|\,c_1\,) \trianglerighteq (X\,|\,c_1 \vee c_2\,)] \\ \wedge \\ [(X\,|\,c_1 \vee c_2\,) \trianglerighteq (X\,|\,c_2\,)] \end{array}\right) \end{array}\right] \forall\,(X, c_1, c_2)\ , \tag{6}$$

which is a special case of (A9) below.

**Axiom 8 (Equiprobability or Infraprobability of Conjoined Contingency):**[6]

$$\left[\begin{array}{c} [(X_2\,|\,c\,) \trianglerighteq (X_1 \wedge X_2\,|\,c\,)] \\ \wedge \\ \left(\begin{array}{c} [(X_2\,|\,c\,) \boxdot (X_1 \wedge X_2\,|\,c\,)] \\ \Leftrightarrow \\ [(c\,|\,c\,) \boxdot (X_1\,|\,X_2 \wedge c\,)] \end{array}\right) \\ \wedge \\ \left(\begin{array}{c} [(X_1\,|\,X_2 \wedge c\,) \vartriangleright (X_1 \wedge X_2\,|\,c\,)] \\ \Leftrightarrow \\ \left(\begin{array}{c} [(c\,|\,c\,) \vartriangleright (X_2\,|\,c\,)] \\ \wedge \\ [(X_1 \wedge X_2\,|\,c\,) \vartriangleright (\overline{c}\,|\,c\,)] \end{array}\right) \end{array}\right) \\ \wedge \\ \left(\begin{array}{c} [(X_1\,|\,X_2 \wedge c\,) \boxdot (X_1 \wedge X_2\,|\,c\,)] \\ \Leftrightarrow \\ \left(\begin{array}{c} [(c\,|\,c\,) \boxdot (X_2\,|\,c\,)] \\ \vee \\ [(X_1 \wedge X_2\,|\,c\,) \boxdot (\overline{c}\,|\,c\,)] \end{array}\right) \end{array}\right) \end{array}\right] \forall\,(X_1, X_2, c) \tag{A8}$$

[6] $(\overline{c}\,|\,c\,) = (c \wedge \overline{c}\,|\,c\,)$; thus, the use of $(\overline{c}\,|\,c\,)$ for minimal plausibility may be viewed as an application of the Law of Non-Contradiction [1, bk 4, ch 3, 1005b15-22]. Keynes and Koopman, despite their misgivings about the general applicability of complete preörderings, concerned themselves primarily with finding conditions for just such application (and, more generally, for use of arithmetic) and moved quickly to introduce a 0 [4, ch X, §4, def III & ch XII, §4, def III][5, §1 fn 3][6]. In Koopman's case, this introduction caused him to include needless complications in his axiomata of Composition and of Decomposition, which conditions will be noted below.

(The necessity claims entail some redundancy in the context of (A6), but simplify expression.)

The first two parts of (A8), that

$$\left[\begin{array}{c}[(X_1\,|\,c) \trianglerighteq (X_1 \wedge X_2\,|\,c)] \\ \wedge \\ \left(\begin{array}{c}[(X_1\,|\,c) \boxdot (X_1 \wedge X_2\,|\,c)] \\ \Leftrightarrow \\ [(c\,|\,c) \boxdot (X_2\,|\,X_1 \wedge c)]\end{array}\right)\end{array}\right] \forall\,(X_1, X_2, c)\ , \tag{7}$$

are jointly a variation on Ockham's Razor. In the case where $X_1 = Y_1 \vee \overline{Y_2}$ and $X_2 = Y_1 \vee Y_2$, noting that

$$\begin{array}{rcl}\big(Y_1 \vee \overline{Y_2}\,|\,(Y_1 \vee Y_2) \wedge c\big) & = & \big((Y_1 \vee \overline{Y_2}) \wedge (Y_1 \vee Y_2)\,|\,(Y_1 \vee Y_2) \wedge c\big) \\ & = & (Y_1\,|\,(Y_1 \wedge c) \vee (Y_2 \wedge c))\end{array}$$

and employing (A7) with (A4), with (A2), and with (A3), one arrives at

$$\left[\begin{array}{c}[(Y_1 \vee Y_2\,|\,c) \trianglerighteq (Y_1\,|\,c)] \\ \wedge \\ \left(\begin{array}{c}[(Y_1 \vee Y_2\,|\,c) \boxdot (Y_1\,|\,c)] \\ \Leftrightarrow \\ [(c\,|\,c) \boxdot (Y_1\,|\,Y_2 \wedge c)]\end{array}\right)\end{array}\right] \forall\,(Y_1, Y_2, c)\ , \tag{8}$$

which conforms to an intuition that relaxing assumptions is Ockhamistic.

The remainder of (A8) asserts that, if $c \Rightarrow X_2$, then $(X_1\,|\,c) = (X_1\,|\,X_2 \wedge c) = (X_1 \wedge X_2\,|\,c)$; and that, otherwise, taking $X_2$ as given imputes greater plausibility (except in the case in which $X_1$ contradicts $X_2$). Arriving at such an imputation by a spurious presumption is of course *begging the question*. Continuing with $X_1 = Y_1 \vee \overline{Y_2}$ and $X_2 = Y_1 \vee Y_2$, one arrives at

$$\left[\begin{array}{c}\left(\begin{array}{c}[(Y_1\,|\,(Y_1 \vee Y_2) \wedge c) \triangleright (Y_1\,|\,c)] \\ \Leftrightarrow \\ \left(\begin{array}{c}[(c\,|\,c) \triangleright (Y_1 \vee Y_2\,|\,c)] \\ \wedge \\ [(Y_1\,|\,c) \triangleright (\bar{c}\,|\,c)]\end{array}\right)\end{array}\right) \\ \wedge \\ \left(\begin{array}{c}[(Y_1\,|\,(Y_1 \vee Y_2) \wedge c) \boxdot (Y_1\,|\,c)] \\ \Leftrightarrow \\ \left(\begin{array}{c}[(c\,|\,c) \boxdot (Y_1 \vee Y_2\,|\,c)] \\ \vee \\ [(Y_1\,|\,c) \boxdot (\bar{c}\,|\,c)]\end{array}\right)\end{array}\right)\end{array}\right] \forall\,(Y_1, Y_2, c)\ , \tag{9}$$

which isn't as elegant as the transformation of (7) into (8).

**Axiom 9 (Disjunctive Presupposition):**

$$\left(\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}[(X\,|\,c_1\,) \trianglerighteq (Y\,|\,c_2\,)]\\ \wedge \\ [(X\,|\,c_1\,) \trianglerighteq (Y\,|\,c_3\,)]\end{array}\right)\\ \Rightarrow \\ [(X\,|\,c_1\,) \trianglerighteq (Y\,|\,c_2 \vee c_3\,)]\end{array}\right]\\ \wedge \\ \left[\begin{array}{c}\left(\begin{array}{c}[(Y\,|\,c_2\,) \trianglerighteq (X\,|\,c_1\,)]\\ \wedge \\ [(Y\,|\,c_3\,) \trianglerighteq (X\,|\,c_1\,)]\end{array}\right)\\ \Rightarrow \\ [(Y\,|\,c_2 \vee c_3\,) \trianglerighteq (X\,|\,c_1\,)]\end{array}\right]\end{array}\right) \forall\,(X, Y, c_1, c_2, c_3) \tag{A9}$$

This principle is not quite just a refactoring of Koopman's axiom of *Alternative Presumption* [5][6],

$$\left[\begin{array}{c}\left(\begin{array}{c}[(X\,|\,c_1\,) \trianglerighteq (Y\,|\,c_2 \wedge d\,)]\\ \wedge \\ \left[(X\,|\,c_1\,) \trianglerighteq \left(Y\,\middle|\,c_2 \wedge \overline{d}\right)\right]\end{array}\right)\\ \Rightarrow \\ [(X\,|\,c_1\,) \trianglerighteq (Y\,|\,c_2\,)]\end{array}\right] \forall\,(X, Y, c_1, c_2, d)\;\;, \tag{10}$$

as nothing for the latter corresponds to what would be the case of $c_2 = c_1$ in the former. But (10) is an immediate implication of (A9).

**Axiom 10 (Generalized Subdivision):**

$$\left[\begin{array}{c}\left(\begin{array}{c}[(X_1 \vee X_2\,|\,c_1\,) \trianglerighteq (Y_1 \vee Y_2\,|\,c_2\,)]\\ \wedge \\ [(X_1 \wedge X_2\,|\,c_1\,) \trianglerighteq (Y_1 \wedge Y_2\,|\,c_2\,)]\\ \wedge \\ [(X_1\,|\,c_1\,) \trianglerighteq (X_2\,|\,c_1\,)]\\ \wedge \\ [(Y_1\,|\,c_2\,) \trianglerighteq (Y_2\,|\,c_2\,)]\end{array}\right)\\ \Rightarrow \\ [(X_1\,|\,c_1\,) \trianglerighteq (Y_2\,|\,c_2\,)]\end{array}\right] \forall \left(\begin{array}{r}X_1, X_2,\\ Y_1, Y_2,\\ c_1, c_2\end{array}\right) \tag{A10}$$

Koopman's presentations of his narrower principle of *Subdivision* use a combination of formal and natural expression [5][6]. Expressed formally and in full, it may be seen to be quite involved.

$$\left[\begin{array}{c}\left(\begin{array}{c}\left[\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}\overline{i=j}\\ \wedge \\ \left[\begin{array}{c}(i,j)\\ \in \\ \{1,2,\ldots,n\}^2\end{array}\right]\end{array}\right)\\ \Rightarrow \\ \left(\overline{X_i \wedge X_j} \wedge \overline{Y_i \wedge Y_j}\right)\end{array}\right]\forall\,(i,j)\\ \wedge \\ \left(\begin{array}{c}\left[\begin{array}{c}(X_{j+1}\,|\,\bigvee_{k=1}^{n} X_k\,)\\ \trianglerighteq \\ (X_j\,|\,\bigvee_{k=1}^{n} X_k\,)\end{array}\right]\\ \Leftarrow \\ \left[j \in \{1,2,\ldots,n-1\}\right]\end{array}\right)\forall j\\ \wedge \\ \left(\begin{array}{c}\left[\begin{array}{c}(Y_{j+1}\,|\,\bigvee_{k=1}^{n} Y_k\,)\\ \trianglerighteq \\ (Y_j\,|\,\bigvee_{k=1}^{n} Y_k\,)\end{array}\right]\\ \Leftarrow \\ \left[j \in \{1,2,\ldots,n-1\}\right]\end{array}\right)\forall j\end{array}\right]\\ \Rightarrow \\ \left[\begin{array}{c}(X_n\,|\,\bigvee_{k=1}^{n} X_k\,)\\ \trianglerighteq \\ (Y_1\,|\,\bigvee_{k=1}^{n} Y_k\,)\end{array}\right]\end{array}\right)\forall\left(\begin{array}{c}\left\{\begin{array}{c}X_1, X_2,\\ \ldots, X_n\end{array}\right\},\\ \left\{\begin{array}{c}Y_1, Y_2,\\ \ldots, Y_n\end{array}\right\}\end{array}\right)\\ \Leftarrow \\ (n \in \mathbb{N}_1)\end{array}\right]\forall n \tag{11}$$

But, (A10) in combination with (A6) tell us that

$$\left[\begin{array}{c}\left(\begin{array}{c}\left[(X\,|\,c_1\,) \trianglerighteq \left(\overline{X}\,|\,c_1\,\right)\right]\\ \wedge \\ \left[(Y\,|\,c_2\,) \trianglerighteq \left(\overline{Y}\,|\,c_2\,\right)\right]\end{array}\right)\\ \Rightarrow \\ \left[(X\,|\,c_1\,) \trianglerighteq \left(\overline{Y}\,|\,c_2\,\right)\right]\end{array}\right]\forall\,(X_1, X_2, c_1, c_2)$$

which provides a base for arriving at (11) inductively by further application of (A10).

**Axiom 11 (Generalized Additivity):**

$$\left[\begin{array}{c}\left(\begin{array}{c}\left[(Y_1 \wedge Y_2\,|\,c_2\,) \trianglerighteq (X_1 \wedge X_2\,|\,c_1\,)\right]\\ \wedge \\ \left[(X_1\,|\,c_1\,) \trianglerighteq (Y_1\,|\,c_2\,)\right]\\ \wedge \\ \left[(X_2\,|\,c_1\,) \trianglerighteq (Y_2\,|\,c_2\,)\right]\end{array}\right)\\ \Rightarrow \\ \left[(X_1 \vee X_2\,|\,c_1\,) \trianglerighteq (Y_1 \vee Y_2\,|\,c_2\,)\right]\end{array}\right]\forall\left(\begin{array}{r}X_1, X_2,\\ Y_1, Y_2,\\ c_1, c_2\end{array}\right) \tag{A11}$$

This axiom is a straight-forward generalization of the popular principle of *Additivity* [2][3].

(A12) and (A13) correspond to Koopman's principles of *Composition* and of *Decomposition* [5][6], but are simpler and more general. So long as Koopman's principles of *Composition* and of *Decomposition* were discernible by intuition, identifying redundancies would be largely irrelevant to to his project. But they have been particularly challenged as insufficiently self-evident [8], and the complexity of their expression could not had favorable effect on that perception.

**Axiom 12 (Generalized Composition):**

$$\left(\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}[(X_1\,|\,c_1\,)\trianglerighteq(Y_1\,|\,c_2\,)]\\ \wedge\\ [(X_2\,|\,X_1\wedge c_1\,)\trianglerighteq(Y_2\,|\,Y_1\wedge c_2\,)]\end{array}\right)\\ \vee\\ \left(\begin{array}{c}[(X_1\,|\,c_1\,)\trianglerighteq(Y_2\,|\,Y_1\wedge c_2\,)]\\ \wedge\\ [(X_2\,|\,X_1\wedge c_1\,)\trianglerighteq(Y_1\,|\,c_2\,)]\end{array}\right)\end{array}\right]\\ \Rightarrow\\ [(X_1\wedge X_2\,|\,c_1\,)\trianglerighteq(Y_1\wedge Y_2\,|\,c_2\,)]\end{array}\right)\forall\left(\begin{array}{r}X_1,X_2,\\ Y_1,Y_2,\\ c_1,c_2\end{array}\right)\qquad\text{(A12)}$$

Koopman's axiom of *Composition* contains two conditions not found in (A12). One is that $(Y_1\wedge Y_2\,|\,c_2\,)$ not have minimal probability; this condition is simply unnecessary. The other is that $(X_1\wedge X_2\,|\,c_1\,)$ also not have minimal probability; if (A7) is accepted, then where this condition is actually needed in (A12) it will be implied in the others. Both of these superfluous conditions are artefacts of using "0" to represent any proposition of minimal probability and of declaring propositions not to be 0 (rather than declaring them to be of greater than minimal probability), which representation obfuscated relations.

**Axiom 13 (Generalized Decomposition):**

$$\left(\begin{array}{c}[(X_1 \wedge X_2 \,|\, c_1\,) \trianglerighteq (Y_1 \wedge Y_2 \,|\, c_2\,)] \\ \Rightarrow \\ \left[\begin{array}{c}\left(\begin{array}{c}[(Y_1 \,|\, c_2\,) \trianglerighteq (X_1 \,|\, c_1\,)] \\ \Rightarrow \\ [(X_2 \,|\, X_1 \wedge c_1\,) \trianglerighteq (Y_2 \,|\, Y_1 \wedge c_2\,)]\end{array}\right) \\ \wedge \\ \left(\begin{array}{c}[(Y_1 \,|\, c_2\,) \trianglerighteq (X_2 \,|\, X_1 \wedge c_1\,)] \\ \Rightarrow \\ [(X_1 \,|\, c_1\,) \trianglerighteq (Y_2 \,|\, Y_1 \wedge c_2\,)]\end{array}\right) \\ \wedge \\ \left(\begin{array}{c}[(Y_2 \,|\, Y_1 \wedge c_2\,) \trianglerighteq (X_1 \,|\, c_1\,)] \\ \Rightarrow \\ [(X_2 \,|\, X_1 \wedge c_1\,) \trianglerighteq (Y_1 \,|\, c_2\,)]\end{array}\right) \\ \wedge \\ \left(\begin{array}{c}[(Y_2 \,|\, Y_1 \wedge c_2\,) \trianglerighteq (X_2 \,|\, X_1 \wedge c_1\,)] \\ \Rightarrow \\ [(X_1 \,|\, c_1\,) \trianglerighteq (Y_1 \,|\, c_2\,)]\end{array}\right)\end{array}\right]\end{array}\right) \forall \left(\begin{array}{r}X_1, X_2, \\ Y_1, Y_2, \\ c_1, c_2\end{array}\right) \tag{A13}$$

Koopman's axiom of *Decomposition* again contains a condition that $(Y_1 \wedge Y_2 \,|\, c_2\,)$ not have minimal probability, which is again an artefact of obfuscation from using "0" to represent any proposition of minimal probability and then declaring propositions not to be 0.

### Impossible Presuppositions

Koopman makes a "tacit assumption" that no presupposition have minimal probability [6]. I do not formally exclude such presuppositions, but take the operationally equivalent position that employing them is like including contradictions amongst premises; anything can follow, and the results have no application.

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