Indifference, Indecision, and Coin-Flipping

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Abstract

This paper operationalizes a non-empty relation as implied if *strict preference* and *indifference* jointly do not completely order the choice set. Specifically, *indecision* is operationalized as a positive preference for delegating choice to a least predictable device.

Keywords: incomplete preferences, indifference, indecision, entropy

JEL Classification: D11

Notice to the Reader: This version of the paper differs in a number of respects from the version published. Most of the differences are not of substance but some merit special notice. In the published version:

1. Formulæ have a different numbering.
2. Relational symbols appear more often in the expression of formulæ.
3. Proposition (24) has been replaced by a less general expression.
4. The union of *binary paralysis* with *identity* has been given a formal symbol '$\overset{\pi}{—}$', as in "$X_1 \overset{\pi}{—} X_2$".

Acknowledgements

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1 Introduction

The foundations of theories of economic choice are often discussed in terms of three relations: *strict preference*, *indifference*, and *weak preference*, the last of which is the union of the first two.

$$\succ \subseteq \{X_1, X_2, \ldots\}^2$$

$$\sim \subseteq \{X_1, X_2, \ldots\}^2$$

$$\succcurlyeq \equiv (\succ \cup \sim)$$

It is usually proposed that these relations completely order the set of possible options, so that an agent either *strictly prefers* any given option to any other given option, or is *indifferent* between them.

$$[(X_1 \succcurlyeq X_2) \vee (X_2 \succcurlyeq X_1)] \; \forall (X_1, X_2)$$

If these relations do not completely order a set, that does not mean that some pair of options has is no relation; rather, its relation is a sort of *complement*.

$$R = \{X_1, X_2, \ldots\}^2 \setminus \{(X_1, X_2) \ni [(X_1 \succcurlyeq X_2) \vee (X_2 \succcurlyeq X_1)]\}$$

But there is some challenge in giving an interesting *empirical content* to the supposition of such a relation.

Given an interpretation of *indifference* as *equal valuation*, a natural candidate for this complement would be some sort of *indecision* about relative valuation, such that the individual was neither prepared to say that one choice were better than another nor that they were equally good. But it is actually not immediately clear how behavior under such indecision would differ from that under *indifference*, except for *utterance*. For example, were an agent told that she would be given X_2 if she did not actively request X_1, then she would end up with the *default* if she were *either indifferent* or undecided. And, while an undecided person may later come to the decision that one was indeed better than another, a *change of mind* is also possible with *indifference*. It does not seem much to matter to the economist whether the agent says “I don't know” or “I don't care”.[1] If such indecision does not produce choices different from those of *equal valuation*, then one might as well interpret *indifference* as the union of the two.

However, this paper will identify two relations distinct from *strict preference* which correspond to meaningfully distinct choice behavior. One of these relations will have some intuïtive correspondence to indecision about relative valuation.

[1] Here, “I don't know and I don't care” is either absurd or elliptical.

2 A Problem of Ordinary Interpretation, and an Observable Distinction

When the question is asked of how an agent makes a choice between two things, X_1 and X_2, between which she is *indifferent*, a stock reply is that she "flips a coin". There are at least two problematic aspects to this reply.

The first is that it quite fails to answer the question asked, but presumes that every choice between X_1 and X_2 may be *replaced* with a choice amongst X_1, X_2, and X_3, where X_3 is a *lottery* between X_1 and X_2. The ability to make such replacements is rarely if ever explicit or implicit in the axiomatic structure.

The second problem is that, in structures meant to explain decision-making under risk (as when the value of a lottery is an expectation of its utility), it is typically an axiom or implication that if an agent is *indifferent* amongst all the possible outcomes of a lottery, then the agent is *indifferent* between the lottery and any one of those outcomes.[2] An algorithm of replacement, however, must have it that

$$X_3 \succ X_1$$

or an infinite sequence of lotteries will be introduced into the choice set, without any choice resulting. We are thus compelled to abandon that algorithm, or to revise our model of decision-making under risk, or both.

In actuality, one observes both:

- occasions where people appear to be *indifferent* amongst X_1, X_2, and some non-trivial lottery in which X_1 and X_2 are the possible outcomes;
- occasions where people will "flip a coin" (often quite literally) to make a choice.

One could wave away this distinction, asserting that one or the other of these behaviors represents economic irrationality; but an *indifference* that *mathematically* precluded a preference for "flipping a coin" has long been accepted as rational (and we should resist the temptation to make the economist's life easier by *ad hoc* redefinition of "rationality"), while there would seem to be a meta-preferential argument for permitting Buridan's ass to be saved by the option of a lottery.

In any event, we have enough on-hand to suggest two theoretical alternatives to *strict preference*, each of which has different implications for observed behavior.

3 Formal Structure

Preliminary

The conceptual foundations of a theory of choice are often expressed principally in terms of *preference relations*. In this paper, however, foundations will be

[2] See expression (54) below.

laid in terms of *choice functions*. The *operationalization* of the classic relations and of any proposed additional relation is in the choices that result; and, while choice functions are imperfectly observable, they are observable less indirectly than are preferences.

However, this paper does not have the same sort of ambitions with which revealed preference theory began,[3] and the axiomata will be different.

General Axiomata

The first two axiomata are essentially definitional:

$$[C(B) \subseteq B] \; \forall B \; ; \tag{1}$$

$$([B = \emptyset] \Leftarrow [C(B) = \emptyset]) \, \forall B \; . \tag{2}$$

The first axiom requires that the choice function $C()$ select a subset of the budget (a feasible set or some subset thereof) B; the second that the only set which is mapped to the empty set is itself the empty set.

The next four axiomata are rationality constraints:

$$[C(B_1 \cup B_2) = C[C(B_1) \cup C(B_2)]] \; \forall (B_1, B_2) \; . \tag{3}$$

The choice set of the union of two budgets is the choice set of the union of each of their choice sets. Amongst other things, this axiom says that choices may be made in a *pair-wise* manner.

$$[([C(B_1) \subseteq B_2] \wedge [B_2 \subseteq B_1]) \Rightarrow [C(B_2) = C(B_1)]] \; \forall (B_1, B_2) \; . \tag{4}$$

If the choice set of a budget B_1 is a subset of a subset B_2, then the choice set of B_2 is that of B_1.

$$[([C(B_1) = B_1] \wedge [B_2 \subseteq B_1]) \Rightarrow [C(B_2) = B_2]] \; \forall (B_1, B_2) \; . \tag{5}$$

If the choice set of a budget is all of that budget, then the choice set of any of its sub-budgets is all of the sub-budget.[4]

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[3] Samuelson, Paul Anthony; "A Note on the Pure Theory of Consumer's Behavior", *Economica* v 51 #17 (1938).

[4] Hereïn, an overscoring of a proposition represents negation.

$$\left[\begin{array}{c} \overline{B_1 \subseteq C(B_1 \cup B_2 \cup B_3)}\,\forall B_3 \\ \vee \\ \left(\begin{array}{c} [B_1 \subseteq C(B_1 \cup B_2 \cup B_4)] \\ \vee \\ \overline{B_2 \subseteq C(B_1 \cup B_2 \cup B_4)} \end{array} \right) \forall B_4 \end{array} \right] \forall (B_1, B_2) \ . \tag{6}$$

Given that two budgets (B_1 and B_2) both appear as sub-budgets in two super-budgets ($B_1 \cup B_2 \cup B_3$ and $B_1 \cup B_2 \cup B_4$), it *cannot* be the case that exactly one of these sub-budgets is part of the choice set for one super-budget ($B_1 \cup B_2 \cup B_m$), but the other sub-budget (with or without that one) is part of the choice set for the other super-budget ($B_1 \cup B_2 \cup B_n$).[5]

Some Definitions

Paralysis exists for a budget B if

$$|C(B)| > 1 \ . \tag{7}$$

Unless the budget has exactly two elements, this concept does not perfectly correspond to

$$C(B) = B \ . \tag{8}$$

which, in fact, is not observable except when B has fewer than three members. (Observation of *paralysis* for each binary subset of B only implies (8) under some assumptions, such as axiom (3).) Since *paralysis* is impossible for any singleton budget; any relation of elements to which *paralysis* immediately corresponds cannot be reflexive.

Strict Preference

$$(X_1 \succ X_2) \stackrel{\text{def}}{=} \overline{\{X_2\} \subseteq C(\{X_1, X_2\})} \ . \tag{9}$$

Given the axiomata, this implies

$$([X_1 \succ X_2] \Leftrightarrow [(\{X_1\} \cup B) \subseteq C(\{X_1, X_2\} \cup B)]\,\forall B)\,\forall (X_1, X_2) \ , \tag{10}$$

[5] This axiom is effectively a generalization to budgets of the Weak Axiom of Revealed Preference; Eliaz and Ok instead relax the axiom in their model of indecision. (Eliaz, Krif, and Efe A. Ok; "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences", *Games and Economic Behavior* v 56 #1 (2006) 61–8.)

and, more specifically,

$$([X_1 \succ X_2] \Leftrightarrow [\{X_1\} = C(\{X_1, X_2\})]) \, \forall (X_1, X_2) \ . \tag{11}$$

Non-Rejection

$$(X_1 \not\prec X_2) \stackrel{\text{def}}{=} [\{X_1\} \subseteq C(\{X_1, X_2\})] \ . \tag{12}$$

(In conventional models, *weak preference*, like *non-rejection* here, is the complement of the inverse of *strict preference.* One could define a relation equivalent to binary *paralysis*, in union with identity, from

$$(X_1 \not\prec X_2) \wedge (X_2 \not\prec X_1)$$

much as *indifference* is often defined in terms of *weak preference.*)

Lotteries

The alternatives to *strict preference* will be defined in terms of lotteries. A lottery will be represented as an unordered n-tuple, enclosed in angle-brackets, of comma-separated dyads,[6]

$$\langle (X_1, p_1), (X_2, p_2), \ldots, (X_n, p_n) \rangle,$$

where the first element of each dyad is a description of the world, and the second element is a real number probability associated with that description.

I presume that the outcome of a lottery can itself be described independently of the lottery, and in that context I'll also assert the following three happy equalities:[7]

$$\left[\begin{array}{c} \left(\begin{array}{c} \langle (X,1), (Y_1,0), (Y_2,0), \ldots, (Y_n,0) \rangle \\ = \\ X \end{array} \right) \forall (Y_1, Y_2, \ldots, Y_n) \\ \Leftarrow \\ (n \in \mathbb{N}_1) \end{array} \right] \forall n \forall X \ ; \tag{13}$$

[6]Hereïn, a simple ordered pair may be distinguished from an open interval by immediate context.

[7]Inclusions for logical quantifiers are hereïn written as explicit conjunctions or implications within propositions.

$$\left(\begin{array}{c}\left[\begin{array}{c}\langle (X,p_1),(X,p_2),(Y,q)\rangle \\ = \\ \langle (X,p_1+p_2),(Y,q)\rangle\end{array}\right] \\ \Leftarrow \\ \left[(p_1,p_2,q)\in [0,1]^3\right]\end{array}\right)\forall (p_1,p_2,q)\,\forall Y\forall X\ ; \tag{14}$$

$$\left(\begin{array}{c}\left[\begin{array}{c}\langle (X,p),(\langle (Y_1,q),(Y_2,1-q)\rangle,1-p)\rangle \\ = \\ \langle (X,p),(Y_1,q-p\cdot q),(Y_2,1-p-q+p\cdot q)\rangle\end{array}\right] \\ \Leftarrow \\ \left[(p\in [0,1])\ \wedge\ (q\in [0,1])\right]\end{array}\right)\forall (p,q)\forall Y\forall X\ . \tag{15}$$

(In some other framework, these equalities might be transformed into rationality constraints.)

Three Further Relations

Equi-indifference obtains between any description and itself, and where there is *paralysis* not only amongst the two principal descriptions, but also among the two and some non-trivial lottery across the two. For observability, *equi-indifference* will be defined thus

$$\begin{array}{c}(X_1\approx X_2) \\ \stackrel{\text{def}}{=} \\ \left[\begin{array}{c}\left[C\left(\{X_1,X_2\}\right)\ =\ \{X_1,X_2\}\right] \\ \wedge \\ \left(\left[\begin{array}{c}\left(\begin{array}{c}[X_1\ =\ X_2] \\ \vee \\ \left[\left|C\left(\{X_1,X_2,\langle (X_1,p),(X_2,1-p)\rangle\}\right)\right|>1\right]\end{array}\right) \\ \wedge \\ \left(\begin{array}{c}\left[\begin{array}{c}C\left(\{X_n,\langle (X_1,p),(X_2,1-p)\rangle\}\right) \\ = \\ \{X_n,\langle (X_1,p),(X_2,1-p)\rangle\}\end{array}\right] \\ \Leftarrow \\ \left[X_n\ \in\ \{X_1,X_2\}\right]\end{array}\right)\end{array}\right] \\ \wedge \\ \left[p\ \in\ (0,1)\right]\end{array}\right)\exists p\end{array}\right]\end{array}\ . \tag{16}$$

But a much simpler expression, (51) below, can be derived by application of axiomata (3) and (6).
The relation of *equi-indifference* will correspond to the attitude towards entropy that obtains in the standard (S)EU models. (The quantifier for p will be made universal in (54) as a result of applying axiom (23) below.) An *equi-indifferent* agent will not choose to flip a coin.

Proto-Undecidedness obtains between any description and itself, and where there is *paralysis* between two items but some non-trivial lottery over the two items is *strictly preferred* to either.

$$(X_1 \text{¡} X_2) \stackrel{\text{def}}{=} \left[\begin{array}{c} [C(\{X_1, X_2\}) \;=\; \{X_1, X_2\}] \\ \wedge \\ \left(\begin{array}{c} \left[\begin{array}{c} C(\{X_1, X_2, \langle (X_1, p), (X_2, 1-p) \rangle \}) \\ = \\ \{\langle (X_1, p), (X_2, 1-p) \rangle \} \end{array} \right] \\ \wedge \\ [p \;\in\; (0,1)] \end{array} \right) \exists p \end{array} \right] . \tag{17}$$

Undecidedness obtains where there is *paralysis* between two items but some non-trivial lottery over the two items is *strictly preferred* to either. Its difference from *proto-undecidedness* is exactly in that *undecidedness* is irreflexive (cannot exist between a thing and itself). Undecidedness can be defined in terms of *proto-undecidedness* and identity,

$$\text{¿} \stackrel{\text{def}}{=} (\text{¡} \setminus \;=) \ , \tag{18}$$

or directly in terms of choice functions,

$$(X_1 \text{¿} X_2) \stackrel{\text{def}}{=} \left[\begin{array}{c} [C(\{X_1, X_2\}) \;=\; \{X_1, X_2\}] \\ \wedge \\ \left(\begin{array}{c} \left[\begin{array}{c} C(\{X_1, X_2, \langle (X_1, p), (X_2, 1-p) \rangle \}) \\ \cap \\ \{X_1, X_2\} \\ = \\ \emptyset \end{array} \right] \\ \wedge \\ [p \;\in\; (0,1)] \end{array} \right) \exists p \end{array} \right] . \tag{19}$$

(For definition (19) of "undecidedness", it doesn't really matter whether p is in an open interval or in a closed interval, as the values of 0 and 1 are otherwise ruled-out anyway.)

Given the different attitude towards entropy that is permitted here, it should be no surprise that it is possible (in fact basically required for some cases) in this model for a lottery to be preferred to either of its outcomes, despite one being preferred to the other:

$$\left[\begin{array}{c} \left[C\left(\{X_1,X_2\}\right)\ =\ \{X_1\}\right] \\ \wedge \\ \left(\begin{array}{c} \left[\begin{array}{c} C\left(\{X_1,X_2,\langle(X_1,p)\,,(X_2,1-p)\rangle\}\right)\ \cap\ \{X_1,X_2\} \\ = \\ \emptyset \end{array}\right] \\ \wedge \\ \left[p\ \in\ (0,1)\right] \end{array}\right)\exists p \end{array}\right]\exists\left(X_1,X_2\right)\ . \tag{20}$$

This odd result must obtain in come cases where an outcome is in fact a further lottery. It could be axiomatically excluded in cases where two outcomes of the explicit lottery are not non-trivial lotteries over the same further outcomes, differing only in their respective probabilities for those outcomes.

Further Starting Propositions

The relations of *equi-indifference* and of *undecidedness* are defined and distinguished in terms of lotteries, and the axiomata about choice functions presented to this point ((1) through (6)) do not resolve significant questions because they say nothing about lotteries *per se*.

For example, in the case of *undecidedness* there are questions of which lotteries are preferred (*qua* choices) to certainties, and of what relations obtain within the set of preferred lotteries. Problems of real-world accuracy and precision come into play here; one probability cannot be distinguished from another that were arbitrarily close. If the set of preferred lotteries were a singleton or otherwise countable, then an agent could not be assured that any lottery actually on offer were truly preferred. And, likewise, in the real world, if there ever were a certainty of any sort, it could not be distinguished from an arbitrarily-close non-trivial lottery; so *paralysis* amongst all non-trivial lotteries must then practically be *equi-indifference* between the trivial lotteries.

While a general theory of decision-making under risk would address these questions and more, such completion will not be delivered here. Rather, only a few propositions, sufficient to resolve more immediate concerns, will be produced. Within the context of a more general theory, some or all of these propositions might be theoremata, but for purposes here they will be treated as axiomata.

Symmetrical-Entropy Neutrality

$$\left[\left(\begin{array}{c}[C\left(\{X_1,X_2\}\right) \;=\; \{X_1,X_2\}]\\ \Rightarrow\\ \left(\left[\begin{array}{c} C\left(\left\{\begin{array}{c}\langle(X_1,p)\,,(X_2,1-p)\rangle\,,\\ \langle(X_1,1-p)\,,(X_2,p)\rangle\end{array}\right\}\right)\\ =\\ \left\{\begin{array}{c}\langle(X_1,p)\,,(X_2,1-p)\rangle\,,\\ \langle(X_1,1-p)\,,(X_2,p)\rangle\end{array}\right\}\end{array}\right]\\ \Leftarrow\\ [p\;\in\;(0,1)]\right)\end{array}\right)\forall p\right]\forall\left(X_1,X_2\right)\ . \tag{21}$$

In cases where $X_1 \;=\; X_2$, (21) would simply follow from lottery identities (13) and (14). In other cases, it proposes that if there is *paralysis* between two outcomes then there will also be *paralysis* between any two lotteries across the outcomes such that the probabilities in one are simply an exchange of the probabilities in the other.

Non-Rejectability of Certainty Implying Non-Rejectability of Probability

$$\left(\begin{array}{c}[\{X_1\}\;\subseteq\;C\left(\{X_1,X_2\}\right)]\\ \Rightarrow\\ \left[\begin{array}{c}\left(\begin{array}{c}\{\langle(X_1,p)\,,(X_2,1\;-\;p)\rangle\}\\ \subseteq\\ C\left[\{\langle(X_1,p)\,,(X_2,1\;-\;p)\rangle\,,X_2\}\right]\end{array}\right)\\ \Leftarrow\\ (p\;\in\;[0,1])\end{array}\right]\forall p\end{array}\right)\forall\left(X_1,X_2\right)\ . \tag{22}$$

If an outcome X_2 is not preferred to an outcome X_1, then the certainty of X_2 is not preferred to any lottery across the two outcomes.

Desirability of Certainty Implying Desirability of Probability

$$\left(\begin{array}{c}\left[\begin{array}{c}\left(\begin{array}{c}C\left[\left\{X_1,\left\langle\left(X_1,q\right),\left(X_2,1\ -\ q\right)\right\rangle\right\}\right]\\ =\\ X_1\end{array}\right)\\ \Leftarrow\\ \left[q\ \in\ [0,1)\right]\end{array}\right]\forall q\\ \Rightarrow\\ \left[\begin{array}{c}\left(\begin{array}{c}C\left[\left\{\left\langle\left(X_1,r\right),\left(X_2,1\ -\ r\right)\right\rangle,X_2\right\}\right]\\ =\\ \left\{\left\langle\left(X_1,r\right),\left(X_2,1\ -\ r\right)\right\rangle\right\}\end{array}\right)\\ \Leftarrow\\ \left(r\ \in\ (0,1]\right)\end{array}\right]\forall r\end{array}\right)\forall\left(X_1,X_2\right)\ . \tag{23}$$

If the certainty of some outcome X_1 is *strictly preferred* to every lottery giving some probability to a rival X_2, then every lottery that gives some probability to X_1 is *strictly preferable* to the certainty of X_2. (Note that this is a weaker claim than one under which increasing probability is preferred.)

Negative Transitivity of Lottery Preference

$$
\left[\begin{array}{c}
\left[\begin{array}{c}
\left(\begin{array}{c}
C\left[\left\{\begin{array}{c}
\left\langle (X_1,p)\,,(X_2,1-p)\right\rangle, \\
\left\langle (X_1,q)\,,(X_2,1-q)\right\rangle
\end{array}\right\}\right] \\
= \\
\left\langle (X_1,p)\,,(X_2,1-p)\right\rangle
\end{array}\right) \\
\wedge \\
\left[(p,q)\ \in\ [0,1]^2\right]\ \wedge\ (q\neq p)
\end{array}\right]\exists\,(p,q) \\
\Rightarrow \\
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(\begin{array}{c}
C\left[\left\{\begin{array}{c}
\left\langle (X_1,r)\,,(X_3,1-r)\right\rangle, \\
\left\langle (X_1,s)\,,(X_3,1-s)\right\rangle
\end{array}\right\}\right] \\
= \\
\left\langle (X_1,r)\,,(X_3,1-r)\right\rangle
\end{array}\right) \\
\vee \\
\left(\begin{array}{c}
C\left[\left\{\begin{array}{c}
\left\langle (X_2,r)\,,(X_3,1-r)\right\rangle, \\
\left\langle (X_2,s)\,,(X_3,1-s)\right\rangle
\end{array}\right\}\right] \\
= \\
\left\langle (X_2,r)\,,(X_3,1-r)\right\rangle
\end{array}\right)
\end{array}\right] \\
\wedge \\
\left[(r,s)\ \in\ [0,1]^2\right]\ \wedge\ (s\neq r)
\end{array}\right)\exists\,(r,s)
\end{array}\right]\forall\,(X_1,X_2,X_3) \tag{24}
$$

If there is some *strictly preferred* lottery across X_1 and X_2, then for any third outcome X_3 there will be a *strictly preferred* lottery across X_1 and X_3, or a *strictly preferred* lottery across X_2 and X_3, or both.[8] In the case of X_1 being *strictly preferred* to X_2 or vice versa, this condition would be met so long as *paralysis* did not obtain both between X_1 and X_3 and between X_2 and X_3, and that's already prohibitted. (See Ordering Theoremata (41) and (44) below.) What is new in this proposition is the claim that when a non-trivial lottery is *strictly preferred* over some lottery (trivial or otherwise) amongst lotteries over X_1 and X_2, then some lottery (trivial or otherwise) will be *strictly preferred* amongst lotteries over X_1 and X_3, or amongst lotteries over X_2 and X_3, or both.

Desirability of Lotteries across Paralyzing Lotteries Given Proto-Undecidedness between Underlying Outcomes

[8] Note that this is a negative transitivity of *strict preference*, not of *weak preference*.

$$\left(\begin{array}{c} \left[\begin{array}{c} [C(\{X_1, X_2\}) \;=\; \{X_1, X_2\}] \\ \wedge \\ \left(\begin{array}{c} \left[\begin{array}{c} C(\{X_1, \langle (X_1, p), (X_2, 1-p) \rangle \}) \\ = \\ \{\langle (X_1, p), (X_2, 1-p) \rangle\} \end{array} \right] \\ \wedge \\ [p \;\in\; (0,1)] \end{array} \right) \exists p \end{array} \right] \\ \Rightarrow \\ \left[\left(\begin{array}{c} \left(\begin{array}{c} C\left[\left\{ \begin{array}{c} \langle (X_1, q), (X_2, 1-q) \rangle, \\ \langle (X_1, r), (X_2, 1-r) \rangle \end{array} \right\}\right] \\ = \\ \left\{ \begin{array}{c} \langle (X_1, q), (X_2, 1-q) \rangle, \\ \langle (X_1, r), (X_2, 1-r) \rangle \end{array} \right\} \end{array} \right) \\ \Rightarrow \\ \left[\begin{array}{c} \left(\begin{array}{c} C\left[\left\{ \begin{array}{c} \langle (X_1, q), (X_2, 1-q) \rangle, \\ \langle (X_1, s), (X_2, 1-s) \rangle \end{array} \right\}\right] \\ = \\ \{\langle (X_1, s), (X_2, 1-s) \rangle\} \end{array} \right) \\ \Leftarrow \\ ([s \;\in\; (q,r)] \;\vee\; [s \;\in\; (r,q)]) \end{array} \right] \exists s \end{array} \right) \\ \Leftarrow \\ \left(\left[(q,r) \;\in\; [0,1]^2 \right] \;\wedge\; [q \;\neq\; r] \right) \end{array} \right] \forall (q,r) \end{array} \right) \forall (X_1, X_2) \;. \tag{25}$$

Again, in cases where $X_1 \;=\; X_2$, (25) holds trivially from the lottery identities (14) and (15). Otherwise, this proposition holds that, if *paralysis* obtains between distinct outcomes X_1 and X_2, and some non-trivial lottery across the two is preferred to X_1, then, *paralysis* between any two lotteries across X_1 and X_2 will imply *proto-undecidedness* between these two lotteries.

Theoremata[9]

Mutual Implication of the Null Set:

[9] *Arithmetic* as such plays a very limited rôle hereïn; these theoremata are mostly a working-out of the *logic* of the prior formal propositions. Hence, the style of proof will perhaps be more familiar to logicians than to most economists.

$$([B = \emptyset] \Leftrightarrow [C(B) = \emptyset])\,\forall B\ . \tag{26}$$

Proof: From (1) and (2). ■

Coupling Theorem:

$$\left(\begin{array}{c} [(B_1 \cup B_2) \subseteq C(B_1 \cup B_2 \cup B_3)]\ \exists B_3 \\ \Rightarrow \\ \left(\begin{array}{c} [B_1 \subseteq C(B_1 \cup B_2 \cup B_4)] \\ \Leftrightarrow \\ [B_2 \subseteq C(B_1 \cup B_2 \cup B_4)] \end{array}\right) \forall B_4 \end{array}\right) \forall (B_1, B_2)\ . \tag{27}$$

Proof: Apply the definition of implication and a Law of DeMorgan to (6),

$$\left[\begin{array}{c} [B_1 \subseteq C(B_1 \cup B_2 \cup B_3)]\ \exists B_3 \\ \Rightarrow \\ \left(\begin{array}{c} [B_2 \subseteq C(B_1 \cup B_2 \cup B_4)] \\ \Rightarrow \\ [B_1 \subseteq C(B_1 \cup B_2 \cup B_4)] \end{array}\right) \forall B_4 \end{array}\right] \forall (B_1, B_2)\ . \tag{28}$$

and note that

$$([(B_1 \cup B_2) \subseteq B_3] \Leftrightarrow [(B_1 \subseteq B_3) \wedge (B_2 \subseteq B_3)])\,\forall (B_1, B_2, B_3)\ .■ \tag{29}$$

Exhaustion of possibilities by non-rejectability:

$$\left(\begin{array}{c} [X_1 \succ X_2] \vee [X_2 \succ X_1] \\ \vee \\ [C(\{X_1, X_2\}) = \{X_1, X_2\}] \end{array}\right) \forall (X_1, X_2)\ . \tag{30}$$

Proof: From (1),

$$\left(\begin{array}{c} [C(\{X_1, X_2\}) = \{X_1\}] \\ \vee \\ [C(\{X_1, X_2\}) = \{X_2\}] \\ \vee \\ [C(\{X_1, X_2\}) = \{X_1, X_2\}] \end{array}\right) \forall (X_1, X_2)\ .■ \tag{31}$$

Transitivity of Strict Preference:

$$\begin{pmatrix} [(X_1 \succ X_2) \wedge (X_2 \succ X_3)] \\ \Rightarrow \\ [X_1 \succ X_3] \end{pmatrix} \forall (X_1, X_2, X_3) \ . \tag{32}$$

Proof (by contradiction)*:* From (3),

$$C(\{X_1, X_2, X_3\}) = C[C(\{X_1\}) \cup C(\{X_2, X_3\})] = C(\{X_1, X_2\}) = \{X_1\} .$$

But, from (6),

$$\left[\begin{pmatrix} [\{X_3\} \subseteq C(\{X_1, X_3\})] \\ \wedge \\ [\{X_1\} \subseteq C(\{X_1, X_2, X_3\})] \end{pmatrix} \begin{matrix} \\ \Rightarrow \\ [\{X_3\} \subseteq C(\{X_1, X_2, X_3\})] \end{matrix} \right] \forall (X_1, X_2, X_3) \ . \tag{33}$$

Acyclicity of Strict Preference:

$$\left[(X_1 \succ X_2) \wedge (X_2 \succ X_3) \Rightarrow \overline{X_3 \succ X_1} \right] \forall (X_1, X_2, X_3) \ . \tag{34}$$

Proof (by contradiction)*:* From axiom (3),

$$\left(\left[\begin{array}{c}[C(\{X_1,X_2\}) \;=\; \{X_1\}] \\ \wedge \\ [C(\{X_2,X_3\}) \;=\; \{X_2\}] \\ \wedge \\ [C(\{X_1,X_3\}) \;=\; \{X_3\}] \\ \wedge \\ \begin{pmatrix} C\left[C(\{X_1\}) \,\cup\, C(\{X_2,X_3\})\right] \\ = \\ C\left[C(\{X_1,X_3\}) \,\cup\, C(\{X_2\})\right] \end{pmatrix} \forall\,(X_1,X_2,X_3) \\ \wedge \\ \begin{pmatrix} C\left[C(\{X_1,X_3\}) \,\cup\, C(\{X_2\})\right] \\ = \\ C\left[C(\{X_1,X_2\}) \,\cup\, C(\{X_3\})\right] \end{pmatrix} \forall\,(X_1,X_2,X_3) \end{array}\right] \\ \Rightarrow \\ [(\{X_1\} \;=\; \{X_2\}) \;\wedge\; (\{X_2\} \;=\; \{X_3\})]\right) \forall\,(X_1,X_2,X_3) \;\; . \tag{35}$$

Symmetry of $C(\{X_i,X_j\}) \;=\; \{X_i,X_j\}$*:*

$$\begin{pmatrix} [C(\{X_1,X_2\}) \;=\; \{X_1,X_2\}] \\ \Rightarrow \\ [C(\{X_2,X_1\}) \;=\; \{X_2,X_1\}] \end{pmatrix} \forall\,(X_1,X_2) \;\; . \tag{36}$$

Proof:

$$(\{X_2,X_1\} \;\equiv\; \{X_1,X_2\}) \;\forall\,(X_1,X_2) \;\; .\blacksquare \tag{37}$$

Transitivity of Binary Paralysis:

$$\left[\begin{array}{c}\begin{pmatrix}[C(\{X_1,X_2\}) \;=\; \{X_1,X_2\}] \\ \wedge \\ [C(\{X_2,X_3\}) \;=\; \{X_2,X_3\}]\end{pmatrix} \\ \Rightarrow \\ [C(\{X_1,X_3\}) \;=\; \{X_1,X_3\}]\end{array}\right] \forall\,(X_1,X_2,X_3) \;\; . \tag{38}$$

Proof: Trivially true when $X_1 = X_2$ and when $X_2 = X_3$. Otherwise, from (6),

$$\begin{bmatrix} \begin{pmatrix} [C(\{X_1, X_2\}) = \{X_1, X_2\}] \\ \wedge \\ [C(\{X_2, X_3\}) = \{X_2, X_3\}] \end{pmatrix} \\ \Rightarrow \\ [C(\{X_1, X_2, X_3\}) = \{X_1, X_2, X_3\}] \end{bmatrix} \forall (X_1, X_2, X_3) \ . \tag{39}$$

And, from (1), (2), and (6),

$$\begin{pmatrix} [C(\{X_1, X_2, X_3\}) = \{X_1, X_2, X_3\}] \\ \Rightarrow \\ [C(\{X_1, X_3\}) = \{X_1, X_3\}] \end{pmatrix} \forall (X_1, X_2, X_3) \ . \blacksquare \tag{40}$$

Ordering Theorem (41)*:*

$$\begin{bmatrix} ([C(\{X_1, X_2\}) = \{X_1, X_2\}] \wedge [X_2 \succ X_3]) \\ \Rightarrow \\ (X_1 \succ X_3) \end{bmatrix} \forall (X_1, X_2, X_2) \ . \tag{41}$$

If $X_1 = X_2$ or if an agent is paralyzed between X_1 and X_2, and if the agent *strictly prefers* X_2 to X_3, then she *strictly prefers* X_1 to X_3.

Proof (by contradiction)*:* Trivially true when $X_1 = X_2$. Otherwise, from (3),

$$\begin{pmatrix} C[C(\{X_1\}) \cup C(\{X_2, X_3\})] \\ = \\ C[C(\{X_1, X_3\}) \cup C(\{X_2\})] \end{pmatrix} \forall (X_1, X_2, X_3) \ . \tag{42}$$

Hence

$$\begin{bmatrix} (X_2 \succ X_3) \\ \Rightarrow \\ (C(\{X_1\} \cup \{X_2\}) = C[C(\{X_1, X_3\}) \cup \{X_2\}]) \end{bmatrix} \forall (X_1, X_2, X_2) \ . \tag{43}$$

Hence

$$[\{X_3\} \ = \ C(\{X_1, X_3\})] \ \Rightarrow \ [C(\{X_1, X_2\}) \ = \ \{X_2\}] \ .$$

Further, binary since *paralysis* is transitive (theorem (38)),

$$\begin{matrix} ([C(\{X_1, X_2\}) \ = \ \{X_1, X_2\}] \ \wedge \ [C(\{X_1, X_3\}) \ = \ \{X_1, X_3\}]) \\ \Rightarrow \\ [C(\{X_2, X_3\}) \ = \ \{X_2, X_3\}] \end{matrix} \quad .\blacksquare$$

Ordering Theorem (44)*:*

$$\begin{bmatrix} ([X_1 \ \succ \ X_2] \ \wedge \ [C(\{X_2, X_3\}) \ = \ \{X_2, X_3\}]) \\ \Rightarrow \\ (X_1 \ \succ \ X_3) \end{bmatrix} \forall (X_1, X_2, X_3) \ . \tag{44}$$

If an agent *strictly prefers* X_1 to X_2 and is paralyzed between X_2 and X_3 or if he *strictly prefers* X_1 to X_2 and $X_2 \ = \ X_3$, then she *strictly prefers* X_1 to X_3.

Proof (by contradiction)*:* Trivially true when $X_2 \ = \ X_3$. Otherwise, from (3),

$$\begin{pmatrix} C[C(\{X_2\}) \ \cup \ C(\{X_1, X_3\})] \\ = \\ C[C(\{X_1, X_2\}) \ \cup \ C(\{X_3\})] \end{pmatrix} \forall (X_1, X_2, X_3) \ . \tag{45}$$

Hence

$$\begin{bmatrix} (X_1 \ \succ \ X_2) \\ \Rightarrow \\ (C(\{X_1\} \ \cup \ \{X_3\}) \ = \ C[C(\{X_1, X_3\}) \ \cup \ \{X_2\}]) \end{bmatrix} \forall (X_1, X_2, X_2) \ . \tag{46}$$

Hence

$$[\{X_3\} \ = \ C(\{X_1, X_3\})] \ \Rightarrow \ [C(\{X_2, X_3\}) \ = \ \{X_3\}] \ .$$

Further, since binary *paralysis* is transitive (theorem (38)),

$$\begin{matrix} ([C(\{X_1, X_3\}) \ = \ \{X_1, X_3\}] \ \wedge \ [C(\{X_2, X_3\}) \ = \ \{X_2, X_3\}]) \\ \Rightarrow \\ [C(\{X_1, X_2\}) \ = \ \{X_1, X_2\}] \end{matrix} \quad .\blacksquare$$

Ordering Theorem (47)*:*

$$\left[\begin{pmatrix}[(X_1 \succ X_2) \wedge (X_2 \approx X_3)] \\ \vee \\ [(X_1 \succ X_2) \wedge (X_2 \text{ ¿ } X_3)] \\ \vee \\ [(X_1 \approx X_2) \wedge (X_2 \succ X_3)] \\ \vee \\ [(X_1 \text{ ¿ } X_2) \wedge (X_2 \succ X_3)]\end{pmatrix} \Rightarrow (X_1 \succ X_3)\right] \forall (X_1, X_2, X_3) \ . \quad (47)$$

Proof: From (41) and (44). ■

Reflexivity of equi-indifference:

$$(X_1 \approx X_1) \ \forall (X_1) \ . \quad (48)$$

Proof: By inspection. ■

Symmetry of Equi-Indifference and of Undecidedness:

$$\begin{pmatrix}[(X_1 \approx X_2) \Leftrightarrow (X_2 \approx X_1)] \\ \wedge \\ [(X_1 \text{ ¿ } X_2) \Leftrightarrow (X_2 \text{ ¿ } X_1)]\end{pmatrix} \forall (X_1, X_2) \ . \quad (49)$$

Proof: By inspection. ■

Simplification of Equi-Indifference: From axiom (3),

$$\left[\begin{matrix}(X_1 \approx X_2) \\ \Leftrightarrow \\ \left(\begin{matrix}[C(\{X_1, X_2\}) = \{X_1, X_2\}] \\ \wedge \\ \left[\begin{matrix}\left(\begin{matrix}\left[\begin{matrix}C(\{X_n, \langle (X_1, p), (X_2, 1-p)\rangle\}) \\ = \\ \{X_n, \langle (X_1, p), (X_2, 1-p)\rangle\}\end{matrix}\right] \\ \Leftarrow \\ [X_n \in \{X_1, X_2\}]\end{matrix}\right) \\ \Leftarrow \\ [p \in (0,1)]\end{matrix}\right] \exists p\end{matrix}\right)\end{matrix}\right] \forall (X_1, X_2) \quad (50)$$

and, from (28),

$$\left[\begin{array}{c}(X_1 \approx X_2) \\ \Leftrightarrow \\ \left(\begin{array}{c}\left[\begin{array}{c}C\left(\left\{X_1, X_2, \left\langle (X_1, p), (X_2, 1-p)\right\rangle\right\}\right) \\ = \\ \left\{X_1, X_2, \left\langle (X_1, p), (X_2, 1-p)\right\rangle\right\}\end{array}\right] \\ \wedge \\ (p \in [0,1])\end{array}\right) \exists p\end{array}\right] \forall (X_1, X_2) \;.\blacksquare \tag{51}$$

Paralysis Implying Non-rejection of a Non-Trivial Lottery:

$$\left[\begin{array}{c}(X_1 = X_2) \\ \vee \\ \left(\begin{array}{c}\left[\begin{array}{c}C\left(\left\{X_1, X_2, \left\langle (X_1, p), (X_2, 1-p)\right\rangle\right\}\right) \\ \neq \\ \left\{X_1, X_2\right\}\end{array}\right] \\ \Leftarrow \\ [p \in (0,1)]\end{array}\right) \forall p\end{array}\right] \forall (X_1, X_2) \;. \tag{52}$$

Proof: From (22). ■

Joint Exhaustion by Strict Preference, Equi-Indifference, and Undecidedness:

$$[(X_1 \succ X_2) \vee (X_1 \approx X_2) \vee (X_1 ¿ X_2)] \forall (X_1, X_2) \;. \tag{53}$$

Proof: The definitions themselves exhaust all possibilities not excluded by (53). ■

Universal Entropy-Neutrality of Equi-Indifference:

$$\left(\begin{array}{c}(X_1 \approx X_2) \\ \Leftrightarrow \\ \left[\begin{array}{c}\left(\begin{array}{c}C\left[\left\{X_1, X_2, \left\langle (X_1, p), (X_2, 1-p)\right\rangle\right\}\right] \\ = \\ \left\{X_1, X_2, \left\langle (X_1, p), (X_2, 1-p)\right\rangle\right\}\end{array}\right) \\ \Leftarrow \\ \{p \in [0,1]\}\end{array}\right] \forall p\end{array}\right) \forall (X_1, X_2) \;. \tag{54}$$

Proof (by contradiction)*:* If there is a set of lotteries across X_1 and X_2 that are *strictly preferred* to each, then this set has a maximal subset of lotteries *strictly preferable* to all other lotteries across these two outcomes.

$$\left\{\begin{array}{c} \langle (X_1, p), (X_2, 1 - p) \rangle \\ \ni \\ \left(\begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} \{\langle (X_1, p), (X_2, 1 - p) \rangle\} \\ \subseteq \\ C\left[\{\langle (X_1, p), (X_2, 1 - p) \rangle, \langle (X_1, q), (X_2, 1 - q) \rangle\}\right] \end{array} \right) \\ \Leftarrow \\ [q \in (0,1)] \end{array} \right] \forall q \\ \wedge \\ \left[\begin{array}{c} \left(\begin{array}{c} \{\langle (X_1, p), (X_2, 1 - p) \rangle\} \\ = \\ C\left[\{\langle (X_1, p), (X_2, 1 - p) \rangle, \langle (X_1, q), (X_2, 1 - q) \rangle\}\right] \end{array} \right) \\ \wedge \\ [q \in (0,1)] \end{array} \right] \exists q \end{array} \right) \end{array}\right\} . \quad (55)$$

Under (15), every lottery other than any specific member of this maximal set is a compound lottery across that *strictly preferred* lottery and either X_1 or X_2. Under (23), the other non-trivial lotteries amongst these are all then *strictly preferred* to X_1 or to X_2, as of course are the maximal lotteries.

$$\left(\begin{array}{c} \left[\begin{array}{c} \left(\begin{array}{c} \{\langle (X_1, p), (X_2, 1 - p) \rangle\} \\ = \\ C\left[\{X_1, \langle (X_1, p), (X_2, 1 - p) \rangle\}\right] \end{array} \right) \\ \vee \\ \left(\begin{array}{c} \{\langle (X_1, p), (X_2, 1 - p) \rangle\} \\ = \\ C\left[\{X_2, \langle (X_1, p), (X_2, 1 - p) \rangle\}\right] \end{array} \right) \end{array} \right] \\ \Leftarrow \\ [p \in (0,1)] \end{array} \right) \forall p \; , \quad (56)$$

ruling out *equi-indifference.* ■

Desirability of Non-Trivial Lotteries Given Proto-Undecidedness:

$$\left(\begin{array}{c}\left[\begin{array}{c}[C\left(\{X_1, X_2\}\right) \; = \; \{X_1, X_2\}] \\ \wedge \\ \left(\begin{array}{c}\left[\begin{array}{c}C\left(\{X_1, \langle (X_1, p), (X_2, 1-p)\rangle\}\right) \\ = \\ \{\langle (X_1, p), (X_2, 1-p)\rangle\}\end{array}\right] \\ \wedge \\ [p \; \in \; (0,1)]\end{array}\right)\exists p\end{array}\right] \\ \Rightarrow \\ \left(\begin{array}{c}\left[\begin{array}{c}C\left(\{X_1, \langle (X_1, p), (X_2, 1-p)\rangle\}\right) \\ = \\ \{\langle (X_1, p), (X_2, 1-p)\rangle\}\end{array}\right] \\ \wedge \\ [p \; \in \; (0,1)]\end{array}\right)\forall p\end{array}\right)\forall (X_1, X_2) \; . \tag{57}$$

When $X_1 \;\neq\; X_2$, this proposition holds that, if *paralysis* obtains between distinct outcomes X_1 and X_2, and some non-trivial lottery across the two is preferred to X_1, then all non-trivial lotteries are *strictly preferred* to X_1. (Of course, by symmetry, this would imply that they were *strictly preferred* also to X_2.)

Proof (by contradiction)*:* From the lottery identities (14) and (15) when $X_1 = X_2$ Otherwise, in the context of (53), when *paralysis* obtains between outcomes, either it must obtain between each outcome and any non-trivial lottery across the outcomes, or that non-trival lottery must be *strictly preferred* to either outcome. From theorem (54), if there is any non-trivial lottery which is not *strictly preferred* then no non-trivial lottery is *strictly preferred*, which would contradict definition (17). ■

Mutual Exclusivity of Equi-Indifference and Undecidedness:

$$\overline{(X_1 \;\approx\; X_2) \;\wedge\; (X_1 \text{ ¿ } X_2)}\,\forall (X_1, X_2) \; . \tag{58}$$

Proof: Trivially from definition (18) or (19) and theorem (54). ■

Transitivity of Equi-Indifference:

$$[([X_1 \;\approx\; X_2] \;\wedge\; [X_2 \;\approx\; X_3]) \;\Rightarrow\; (X_1 \;\approx\; X_3)]\,\forall (X_1, X_2, X_3) \; . \tag{59}$$

Proof: Apply DeMorgan's Law and axiomata (1) and (2) to proposition (24). ■

Conjunction of Equi-Indifference with Undecidedness:

$$[([X_1 \;\approx\; X_2] \;\wedge\; [X_2 \text{ ¿ } X_3]) \;\Rightarrow\; (X_1 \text{ ¿ } X_3)] \;\forall\, (X_1, X_2, X_3) \;\; . \tag{60}$$

Proof (by contradiction)*:* Trivially true when $X_1 \;=\; X_2$. Otherwise, since *paralysis* is symmetrical (theorem (36)) and transitive (theorem (38)),

$$C\,(\{X_1, X_3\}) \;=\; \{X_1, X_3\} \;\; . \tag{61}$$

So, under (53),

$$(X_1 \;\approx\; X_3) \;\vee\; (X_1 \text{ ¿ } X_3) \;\; . \tag{62}$$

But, under symmetry and transitivity of *equi-indifference* (theoremata (49) and (59)), $(X_1 \;\approx\; X_3)$ would contradict $(X_1 \text{ ¿ } X_2)$. ■

Intransitivity of Undecidedness:

$$\overline{([(X_1 \text{ ¿ } X_2) \;\wedge\; (X_2 \text{ ¿ } X_3)] \;\Rightarrow\; [X_1 \text{ ¿ } X_3]) \;\forall\, (X_1, X_2, X_3)} \;\; . \tag{63}$$

Proof (by counterexample)*:* Consider the case $X_1 \;=\; X_3$. Undecidedness is irreflexive by definition. ■

Desirability of Entropy under Proto-Undecidedness:

$$\left[\begin{array}{c}
\left[\begin{array}{c}
[C\,(\{X_1, X_2\}) \;=\; \{X_1, X_2\}] \\
\wedge \\
\left(\begin{array}{c}
\left[\begin{array}{c}
C\,(\{X_1, X_2, \langle (X_1, q)\,, (X_2, 1-q)\rangle\}) \\
= \\
\{\langle (X_1, q)\,, (X_2, 1-q)\rangle\}
\end{array}\right] \\
\wedge \\
[q \;\in\; (0,1)]
\end{array}\right) \exists q
\end{array}\right] \\
\Rightarrow \\
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(\begin{array}{c}
C\left[\left\{\begin{array}{c}
\langle (X_1, p)\,, (X_2, 1-p)\rangle\,, \\
\langle (X_1, r)\,, (X_2, 1-r)\rangle
\end{array}\right\}\right] \\
= \\
\langle (X_1, r)\,, (X_2, 1-r)\rangle
\end{array}\right) \\
\wedge \\
([r \;\in\; (p, 1-p)] \;\vee\; [r \;\in\; (1-p, p)])
\end{array}\right] \exists r \\
\Leftarrow \\
(p \;\in\; [0,1])
\end{array}\right) \forall p
\end{array}\right] \forall\, (X_1, X_2) \tag{64}$$

In cases where $X_1 \neq X_2$, (64) proposes that if there is *undecidedness* between two outcomes then one lottery between the two will be *strictly preferred* to another if the probabilities assigned to each in the former are closer together than those in the latter.

Proof: In cases where $X_1 = X_2$, (64) would simply follow from lottery identities (13) and (14). Beyond that, it is little more than a consolidation of (57) and (25) in the context of (21).[10] ■

Optimality, under Undecidedness, of a "Fair Coin":

$$\left[\begin{array}{c} (X_1 \text{ ¿ } X_2) \\ \Rightarrow \\ \left(\begin{array}{c} \left([|p-.5| < |q-.5|] \wedge \left[(p,q) \in [0,1]^2\right]\right) \\ \Rightarrow \\ \left[\begin{array}{c} \langle (X_1,p),(X_2,1-p)\rangle \\ \succ \\ \langle (X_1,q),(X_2,1-q)\rangle \end{array}\right] \end{array}\right) \forall (p,q) \end{array}\right] \forall (X_1,X_2) \; . \tag{65}$$

Proof: This proposition follows from (64). ■

Summary

This model has what can be seen as three basic relations, *strict preference*, *equi-indifference*, and *undecidedness*.

Strict preference is essentially the familiar relation of standard choice theory. It interacts both with *equi-indifference* and with *undecidedness* much as *strict preference* does with *indifference* in that standard model. (See especially theorem (47).)

Equi-indifference, like *indifference* in the standard model, is an equivalence relation. If two outcomes are in a given equivalence set, then so are all lotteries across those outcomes, much as an expected utility model would place them.

[10] Proposition (25) and theorem (64) will perhaps seem more plausible if it is noted that, in the context of (21) and of the lottery identity (15), (64) is marginally equivalent to

$$\left[\begin{array}{c} (X_1 \text{ ¡ } X_2) \\ \Rightarrow \\ \left(\begin{array}{c} \left[\langle (X_1,p),(X_2,1-p)\rangle \text{ ¡ } \langle (X_1,1-p),(X_2 p)\rangle\right] \\ \Leftarrow \\ (p \in [0,1]) \end{array}\right) \forall p \end{array}\right] \forall (X_1,X_2) \, .$$

The gist of which is that if an agent is *proto-undecided* between two choices, then she is *proto-undecided* between the choice of a 30-70 lottery across the two and a 70-30 lottery, of a 40-60 lottery and a 60-40 lottery, &c. (A 50-50 lottery, however, is its own such complement, and the cardinality of the set of a 50-50 lottery with itself is just 1.)

But *equi-indifference* and *strict preference* do not jointly provide a complete ordering of the outcomes.

Undecidedness obtains amongst all remaining outcomes. *Undecidedness* is symmetric, but irreflexive and intransitive. When *undecidedness* obtains between two outcomes, then the agent will choose any non-trivial lottery across the two before simply selecting either, and *strictly prefers* lotteries with less bias, so that a "fair coin" is seen as the best means of selecting an outcome.

4 Discussion

Significance of the Model

The model functions as a sort of proof of concept for an operationalization of preferences as an incomplete preördering (by the union of *strict preference* with *equi-indifference*). Differences, beyond utterance, are observable between *paralysis* which is ended by inclusion of an option of a "coin flip" and that which is not. Plausible propositions imply intuïtively appealing properties to relations defined in terms of these observable behaviors, as well as other properties which are themselves at least plausible.

Theorists and teachers have tended to treat decision-making under certainty as self-contained, if none-the-less a special case. This model may dissolve that containment, as the two alternatives to *strict preference* are distinguished by reference to choices whose outcomes are uncertain. However, the union of those two alternatives (which is also a union of binary *paralysis* and identity) exhibits the classical properties of *indifference* so long as certainty obtains.

The principal significance for theory, then, should be sought in what the distinction would require of the more general theory of decision-making, where certainty is not presumed.

Alternatives Conceptions of a 'Third' Relation

Desire for Delay

An alternate conception of a 'third' relation, neither preference nor *indifference*, associates it with a desire to delay decision-making. That notion and the notion of this paper are special cases of a more general association with a desire to defer in some way the decision-making process, in one case to the agent's future self, in the other case to a present exogenous process. Desire for delay has itself been found empirically.[11]

There is at least some challenge in operationalizing a relevant behavioral distinction between this delaying sort of indecision and traditional *indifference*. If an individual cannot decide at one time, but reaches a decision later, this might be because she had not made up her mind, or because she had *changed* her mind. In some case of this indecision, she might be willing to pay a premium

[11] Danan, Eric, and Ziegelmeyer, Anthony; "Are Preferences Complete? An Experimental Measurement of Indecisiveness under Risk", working paper (2006).

to secure a delay (something that she would not do if she were sure than neither alternative were better than the other);[12] but, in other cases, proposed premia might simply exceed her sense of the potential significance of the distinction between the outcomes.[13]

These two notions of additional relations (one entailing *strict preference* for a lottery, the other *strict preference* for a delay) are non-rival in the sense that it is conceivable that some individuals might choose a lottery but not delay, some might choose delay but not a lottery, and some might prefer both to either of the ultimate outcomes.

Intransitivity

Some authors associate indecision with *intransitivity.* The intuïtion is straightforward; if an agent has no *strict preference* between some x and some y, and likewise has no *strict preference* between y and some z, yet has some clear preference between x and z, then it cannot be that there is *indifference* between x and y and between y and z; rather, it must be that the agent is undecided where to place y in an ordering including x and z. The *behavioral* argument against this sort of intransitivity is that such preferences exhibit a *cyclicity*, under which the agent can fall victim to a *money pump.* For example, say that z is *strictly preferred* to x; in that case, the agent will pay a premium to go from x to z, yet apparently would then go from z to y without charge, and likewise from y to x, whereüpon the agent could once again be induced to pay a fee to go from x to z.

Eliaz and Ok provide an attempt to operationalize incompleteness in terms of intransitivity, and to vindicate its rationality.[14]

Towards vindication, they provide two illustrative examples. Unfortunately, each of these involve some individual (in one case a mother of two children, in the other case an agent awarding fellowships) attempting to conform to external preferences which are *cyclic.* In some case, perhaps in a great *many* cases, the sane chose to humor the insane; they even bend to the will of the lunatic; but the choice made in yielding to regulation by another is not, properly, the same choice as that made by the regulator. I acknowledge that it is desirable to model behavior characterized by cyclicity of preferences, but I would not be comfortable with a model that could not find *indecision* except where one can find this sort of irrationality lurking in the background if not in the foreground.

Eliaz and Ok subsequently provide formal definition of an indecision relation

$$\bowtie \stackrel{\text{def}}{=} \left[X^2 \setminus \left(\succsim \cup \succsim^{-1}\right)\right]$$

[12] Of course, one should distinguish between cases in which a decision may be effected at any time within the additional allotment, and those in which the agent is forced to wait until the end of that allotment.

[13] While "flipping a coin" is not perfectly costless either, the marginal cost of declaring a choice that a coin be flipped can be made essentially identical to that of declaring a choice of one of the principal outcomes.

[14] Eliaz, Kfir, and Ok, Efe A.; "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences", *Games and Economic Behavior* v 56 #1 (2006) 61–86.

(where "X" represents the set of objects of choice), where the relation $\succsim$ rationalizes a choice function $c\,(\,)$, implying that

$$(x \sim y) \;\Rightarrow\; [\{x,y\} \;=\; c\,(\{x,y\})]$$

where

$$\sim \overset{\text{def}}{=} \left(\succsim \cap \succsim^{-1}\right) \;.$$

They then *partially* operationalize $\bowtie$ thus

$$(x \bowtie y) \;\Leftarrow\; \left(\begin{array}{c} (\overline{x=y} \wedge [\{x,y\} = c\,(\{x,y\})]) \\ \wedge \\ \left[\begin{array}{c} \left(\overline{x \in S} \wedge \overline{y \in S}\right) \\ \wedge \\ \left(\begin{array}{c} \overline{[x \in c\,(S \cup \{x\})] \Leftrightarrow [y \in c\,(S \cup \{y\})]} \\ \vee \\ \overline{c\,(S \cup \{x\}) \setminus \{x\} \;=\; c\,(S \cup \{y\}) \setminus \{y\}} \end{array}\right) \\ \wedge \\ (|S| \in \mathbb{N}_1) \end{array}\right] \exists S \end{array}\right) \;.$$

The esential idea is exactly that *apparent* cases of

$$z \sim x \sim y \succ z$$

and of

$$z \succ x \sim y \sim z$$

are actually cases of

$$x \bowtie y \;.$$

However, there is then a remarkable difficulty here in identifying specific instances of *indifference*; apparent *indifference* may be an artefact of not having tested against an S that would expose indecision. A distinction between *indifference* and indecision is not really observable, because *indifference* as such is not really observable.

A very similar operationalization was explored by Mandler. Classically, *indifference* is an equivalence relation, and Mandler proposes to distinguish it from indecision empirically as such.

In an earlier paper,[15] Mandler distinguished *revealed* preferences from *underlying, psychological* preferences (as had some previous authors), and argued that the standard argument for completeness applied to the former and not to the latter, while much of the standard argument for transitivity applied to the

[15]Mandler, Michael; "Incomplete preferences and rational intransitivity of choice", *Games and Economic Behavior* v 50 #2 (2005) 255–77.

latter and not to the former. Cyclicity may then be a feature of revealed preferences, but Mandler proposed that they were not an adequate description of behavior (in which case I would say that they were ill-named), and he submitted that an *extended choice function*, allowing for history-dependent choices, could block any *money pump*. (*Cf* Kyburg's response to *Dutch book* arguments in probability theory.[16]) In my opinion, this is a more satisfying defense of the possible rationality of intransitivity than is found in the examples from Eliaz and Ok.

In a more recent paper,[17] Mandler defines

$$(x \sim * y) \stackrel{\text{def}}{=} (\overline{x \succ y} \wedge \overline{y \succ x})$$

the union of which with $\succ$ would of course represent a complete ordering. But if intransitivity holds, then there are two subcases of $\sim *$,

$$(x \sim_B y) \stackrel{\text{def}}{=} ([(x \sim * z) \Leftrightarrow (y \sim * z)] \,\forall z)$$

which is an equivalence relation, and its complement

$$\perp_B \stackrel{\text{def}}{=} (\sim * \setminus \sim_B)$$

which is not.

$$(x \perp_B y) \equiv \left[(x \sim * y) \wedge \overline{(x \sim * z) \Leftrightarrow (y \sim * z)} \exists z\right] .$$

To be explicit: One finds such a z in a case where

$$z \sim * x \sim * y \succ z$$

or where

$$z \succ x \sim * y \sim * z .$$

The order in which these relations are defined tends to obscure essentially the same difficulty in identifying *indifference* as is found in the aforementioned work by Eliaz and Ok.

Attitudes towards Entropy versus Attitudes towards Risk

If proposition (22) does not obtain in the case where there is *paralysis* between the underlying outcomes, this rejection of entropy represents a sort of aversion to gambling not captured by diminishing marginal utility ('risk aversion'). On the other hand, if some non-trivial lottery is always *strictly preferred* to any two less-entropic choices (so that, for example, *equi-indifference* never held in

[16]Kyburg, Henry Ely jr; "Subjective Probability: Criticisms, Reflections and Problems", *Journal of Philosophical Logic* v 7 #1 (1978) 157–80.

[17]Mandler, Michael; "Indifference and incompleteness distinguished by rational trade", *Games and Economic Behavior* v 67 #1 (2009) 300–14.

the case of *paralysis*), then this pattern would represent a sort of attraction to gambling not captured by the standard notion of being risk loving.

The hypothetical case of an agent who is sometimes classically *indifferent* and sometimes prefers either certain outcome to any non-trivially lottery might be dismissed as unreasonable behavior, or it might be that some more robust framework will make such behavior seem sensible.

5 Areas for Possible Future Work

I have presumed that the outcome of a lottery can be described without reference to the lottery, but some might prefer an alternative conceptualization in which a state of the world intrinsically includes the means by which it was effected, so that one who chooses to "flip a coin" has chosen a different ultimate outcome. The model hereïn could accommodate that conceptualization largely by no more than a reïnterpretation of notation, but the lottery equalities (13), (14), and (15) would have to be replaced with *paralysis* claims, and the mathematics would become more awkward. Results should be fundamentally unchanged.

As noted, the additional proposition (21) through (25) function hereïn as axiomata, but it would ultimately be better to derive some or all of them from more primitive assertions. Proposition (25) looks especially like a rabbit pulled from a hat.

Further, though the distinction between *equi-indifference* and *undecidedness* involves decision-making under risk, nothing resembling a more general theory of such decision-making has been presented here.

There is opportunity, then, perhaps to place these propositions on a better footing, or to replace them altogether with a rival set of propositions still compatible with the earlier axiomata and lottery equalities.

The model in this paper has presumed something like the ordinary notion of probability as a measure of some sort. At this time, conceptions of probability as quantified are so widely and firmly embraced as to be taken for the basic concept. But various authors, such as Keynes,[18] have argued that the plausibility associated with a given outcome may not be subject to quantification or even a complete ordering; in other words, that the relative plausibility of two outcomes may be undecided.

Of course, if subjective orderings of outcomes by plausibility are not complete, the incomparability becomes a source of *undecidedness* amongst preferences across the options associated with those outcomes. I believe that it would probably be trivializing to conceptualize all cases of *undecidedness* as obtaining from indecision about the plausibilities of outcomes, but that it would be appropriate to recognize many or most real-world cases as founded in such incomparability of uncertainties.[19]

[18] Keynes, John Maynard; *A Treatise on Probability*, especially Pt I Ch III.

[19] Remarks to me by Anthony C. Gamst, of UCSD, emphasized the importance of these points.

A decision theory which dispensed with the assumption that outcomes could be completely ordered by plausibility, as well as with the assumption that they could be completely ordered by desirability, could more accurately model the decision-making process of real-world economic agents. The creätion of such a model seems challenging, and may entail an enormous loss of tractability, but could be used to identify where-and-why more conventional models should be expected to fail.

6 Conclusion

A bit more than fifty years ago, Savage wrote

> There is some temptation to explore the possibilities of analyzing preferences among acts as a **partial ordering**, that is, in effect, to replace part 1 of the definition of a simple ordering by the very weak proposition $\mathbf{f} \leq \mathbf{f}$, admitting that some pairs of acts are incomparable. This would seem to give expression to introspective sensations of indecision or vacillation, which we may be reluctant to identify with *indifference*. My own conjecture is that it would prove to be a blind alley losing much power and advancing little, if at all, in realism; but only an enthusiastic exploration could shed real light on the question.[20]

(It is characteristic that Savage would acknowledge his conjecture as such, and encourage its testing.)

There is some advancement in reälism in distinguishing between cases where *paralysis* is resolved by adding the option of a lottery and those in which it is not. Until the implications for general theories of decision-making are more fully explored, it will not be clear to what extent the results of descriptive theory would be affected.

The implications for prescriptive economics seem more apparent. Preferences that are not completely ordered are preferences to which no quantification can be fitted at all, let alone uniquely. It may often be possible to fit *intervals* or more complex structures with elements to which some sort of arithmetic may be applied (work by Dubra, Maccheroni, Eliaz, and Ok has explored such ideas[21] [22]), but the case for these being anything more than *proxies* for orderings will be even weaker than that for point-values. To the extent that prescriptive theories rely upon overt or covert assumptions of interpersonally comparable utility, those theories are cast even further into doubt.

[20] Savage, Leonard Jimmie; *The Foundations of Statististics* (1st and 2nd editions) §2.6, final paragraph. (What Savage then called a "partial ordering" would now more typically be called an "incomplete preordering". Savage of course uses "$\leq$" for a relation that corresponds to a union of *strict preference* with *indifference*.)

[21] Dubra, Juan, Maccheroni, Fabio, and Ok, Efe A.; "Expected Utility without the Completeness Axiom", *Journal of Economic Theory* #115 (2004) 118–33.

[22] Eliaz, Kfir, and Ok, Efe A.; "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences", *Games and Economic Behavior* v 56 #1 (2006) 61–86.